International Journal of Theoretical Physics, Vol. 27, No. 5, 1988

Geometric Calculus: A New Computational Tool for Riemannian Geometry

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Received August 22, 1987

We compare geometric calculus applied to Riemannian geometry with Cartan's exterior calculus method. The correspondence between the two methods is clearly established. The results obtained by a package written in an algebraic language and doing general manipulations on multivectors are compared. We see that the geometric calculus is as powerful as exterior calculus.

1. INTRODUCTION

Let

$$ds^2 = g_{ij}\,dx^i\,dx^j \tag{1}$$

be the line element we want to study. This may be written in a nonholonomic base as follows:

$$ds^2 = \eta_{ij}\omega^i\omega^j \tag{2}$$

where η_{ij} is a constant metric and the ω^i are the basis 1-forms that Cartan's calculus takes as starting point. These are defined by

$$\omega^i = h^i{}_j\,dx^j \tag{3}$$

together with the inverse transformations

$$dx^i = h_j^i\omega^j \tag{4}$$

with

$$h^i{}_j h_k^j = \delta^i{}_k \tag{5}$$

Geometric calculus deals in a very similar way with vectors and multivectors. We can introduce the vectors γ^i and e^i corresponding, respectively,

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0020-7748/88/0500-0613$06.00/0

to the differential forms ω^i and dx^i. These vectors are linked by the same relations as the differential forms, i.e.,

$$\gamma^i = h^i{}_j e^j \tag{6}$$

and their inverse

$$e^i = h_j{}^i \gamma^j \tag{7}$$

To the vectors γ^i we can associate "reciprocal" vectors γ_j defined by the relation

$$\gamma_j \cdot \gamma^i = \delta_j^i \tag{8}$$

Similarly, to the vectors e^i are associated the vectors e_j defined by

$$e_j \cdot e^i = \delta_j^i \tag{9}$$

The indices of the vectors γ are raised and lowered by the constant metric η so that

$$\gamma_i = \eta_{ij}\gamma^j \qquad \text{and} \qquad \gamma_i \cdot \gamma_k = \eta_{ij}\gamma^j \cdot \gamma_k = \eta_{ik} \tag{10}$$

Similarly, the indices of the vectors e are lowered and raised by the metric g:

$$e_i = g_{ij}e^j \qquad \text{and} \qquad e_i \cdot e_k = g_{ij}e^j \cdot e_k = g_{ik} \tag{11}$$

Hence we can now write the relations between the γ_i and the e_i as follows:

$$\gamma_i = h_i{}^j e_j \tag{12}$$

and their inverse

$$e_i = h^j{}_i \gamma_j \tag{13}$$

The relations (11), (13), and (10) provide the relation between the metric tensor g_{ij} and the coefficients $h^k{}_i$:

$$g_{ij} = e_i \cdot e_j = h^k{}_i\gamma_k \cdot h^l{}_j\gamma_l = h^k{}_i h^l{}_j \eta_{kl} \tag{14}$$

2. THE EXTERIOR DERIVATIVE AND THE COCURL

Cartan's method starts with the computation of the exterior derivatives of the basis 1-forms ω^i, (3), written in terms of the ω^i by means of the inverse relations (4). This leads to the first Cartan structural equation:

$$d\omega^i = -\tfrac{1}{2} C^i{}_{kl}\omega^k \wedge \omega^l \tag{15}$$

where the $C^i{}_{kl}$ are the structure coefficients.

By contracting with the constant metric η_{ij} one obtains the covariant form

$$d\omega_i = -\tfrac{1}{2} C_{ikl} \omega^k \wedge \omega^l \tag{15'}$$

with $C_{ikl} = \eta_{im} C^m{}_{kl}$.

Similarly, in geometric calculus we start with the introduction of a differential operator **d** having the properties of a vector and defined by

$$\mathbf{d} = e^i \, \partial/\partial x^i \tag{16}$$

This operator is used to compute what Hestenes calls the cocurl of the vectors γ_k, i.e.,

$$\operatorname{cocurl}(\gamma_k) = \mathbf{d} \wedge \gamma_k = -\tfrac{1}{2} C_{ikl} \gamma^k \wedge \gamma^l \tag{17}$$

In this formula, the γ_k are expressed in terms of the vectors e^i and the final result is written in terms of the γ^i in the same way as the $d\omega_i$ were in terms of the ω^i in (15'). The operation "$\wedge$" is the outer product of Hestenes, which is equivalent to the exterior product, so that (17) gives exactly the same result as (15').

Since the operator **d** is a vector, it can be expressed in the base formed by the γ^i by means of the relations (7). We then have

$$\mathbf{D} = h_j{}^i \gamma^j \, \partial/\partial x^i = \gamma^i X_i \tag{18}$$

with

$$X_j = h_j{}^i \, \partial/\partial x^i \tag{19}$$

From (3), (19), and (5) one deduces immediately

$$\begin{aligned} \omega^i(X_j) &= h^i{}_k \, dx^k \, (h_j{}^l \, \partial/\partial x^l) \\ &= h^i{}_k h_j{}^l \delta^k{}_l = h^i{}_k h_j{}^k = \delta^i{}_j \end{aligned} \tag{20}$$

So we see that the coefficients of **D** in the base formed by the γ^j are nothing else than the base vectors X_j dual to the basis 1-forms ω^j. This duality can be expressed in the geometric calculus formalism by

$$X_j = \gamma_j \cdot \mathbf{D} \tag{21}$$

3. CONNECTIONS AND CODERIVATIVE

The second step in Cartan's method consists in the computation of the connection forms

$$\omega^i{}_j = \Gamma^i{}_{jk} \omega^k \tag{22}$$

or the covariant ones

$$\omega_{ij} = \Gamma_{(i)jk}\omega^k$$

with

$$\Gamma_{(i)jk} = \eta_{il}\Gamma^l{}_{jk} \tag{23}$$

The expressions of $\Gamma_{(i)jk}$ in terms of the C_{ijk} is

$$\Gamma_{(i)jk} = \tfrac{1}{2}(C_{kij} + C_{jik} - C_{ijk}) \tag{24}$$

In geometric calculus the second step is the computation of the 2-vectors $\omega(\gamma_k)$ defined by

$$\begin{aligned}\omega(\gamma_k) &= \tfrac{1}{2}(\gamma^j \wedge \mathbf{d} \wedge \gamma_j) \cdot \gamma_k - \mathbf{d} \wedge \gamma_k \\ &= \tfrac{1}{2}\,\mathrm{TRI} \cdot \gamma_k - \mathrm{cocurl}(\gamma_k)\end{aligned} \tag{25}$$

with

$$\mathrm{TRI} = \gamma^j \wedge \mathbf{d} \wedge \gamma_j = \gamma^j \wedge \mathrm{cocurl}(\gamma_j) \tag{26}$$

$\mathrm{cocurl}(\gamma_j)$ is a 2-vector expressed in terms of $\gamma^m \wedge \gamma^n$ so TRI is a 3-vector expressed in the $\gamma^i \wedge \gamma^j \wedge \gamma^k$ and TRI. γ_k is a 2-vector expressed in the same base as the cocurl. Hence $\omega(\gamma_k)$ is a 2-vector of the form

$$\omega(\gamma_k) = \tfrac{1}{2}\omega_{ij}(\gamma_k)\gamma^i \wedge \gamma^j \tag{27}$$

The meaning of the coefficients $\omega_{ij}(\gamma_k)$ in (27) is easily found by comparing the formula for the coderiative given by Hestenes and Sobcszyk (1984):

$$\delta_a v = d_a v + \omega(a) \cdot v \tag{28}$$

with the classical formula giving the covariant derivative of a differential 1-form v along the vector a.

In (28), $\omega(a)$ is a 2-vector-valued function, which is linear in its argument a, so we have

$$\omega(a) = \omega(a^k\gamma_k) = a^k\omega(\gamma_k) = \tfrac{1}{2}a^k\omega_{ij}(\gamma_k)\gamma^i \wedge \gamma^j \tag{29}$$

The second term of (28) is easily computed and we find

$$\omega(a) \cdot v = \omega_{ij}(a)v^j\gamma^i = a^k\omega_{ij}(\gamma_k)v^j\gamma^i \tag{30}$$

In classical differential geometry the covariant derivative of the form $v = v_k\omega^k$ along the vector $a = a^kX_k$ is given by

$$\nabla_a v = a^k v_{k,i}\omega^i - \Gamma^j{}_{ik}a^k v_j\omega^i$$

or

$$\nabla_a v = a^k v_{k,i}\omega^i - \Gamma_{(j)ik}a^k v^j\omega^i \tag{31}$$

where $v_{k,i} = X_i(v_k)$.

Comparison of (30) with the second term of (31) leads to the following fundamental relation:

$$\omega_{ij}(\gamma_k) = \Gamma_{(i)jk} \tag{32}$$

Remark. The $\Gamma_{(i)jk}$ defined by (24) are antisymmetric on the indices i and j.

The relation (32) shows that the connection coefficients $\Gamma_{(i)jk}$ appearing in Cartan's method are nothing else than the coefficients $\omega_{ij}(\gamma_k)$ of the 2-vector $\omega(\gamma_k)$ given by the formula (27). Hence the computation of one of the $\omega(\gamma_k)$ by (26) gives all the $\Gamma_{(i)jk}$ corresponding to the chosen index k.

4. SECOND CARTAN STRUCTURAL EQUATION AND CURVATURE

The second Cartan structural equation is given by

$$\Theta^i{}_j = d\omega^i{}_j + \omega^i{}_l \wedge \omega^l{}_j \tag{33}$$

or in covariant form

$$\Theta_{ij} = d\omega_{ij} + \omega_{il} \wedge \omega_{mj}\eta^{ml} \tag{33$'$}$$

The components of the Riemann tensor $R^i{}_{jkl}$ or R_{ijkl} appear as the coefficients of the curvature 2-form $\Theta^i{}_j$ or Θ_{ij}, respectively, as seen from the relations

$$\Theta^i{}_j = \tfrac{1}{2} R^i{}_{jkl}\omega^k \wedge \omega^l$$

or

$$\Theta_{ij} = \tfrac{1}{2} R_{ijkl}\omega^k \wedge \omega^l \tag{34}$$

In geometric calculus we introduce a curvature 2-vector defined by

$$R(a \wedge b) = d_a\omega(b) - d_b\omega(a) + \omega(a) \times \omega(b) \tag{35}$$

where the quantities $\omega(a)$ are defined by (29), d_a is what Hestenes calls the "fiducial derivative," and $\times$ is the "cross product" defined by

$$A \times B = 1/2(AB - BA)$$

i.e., the antisymmetrization of the geometric product (Hestenes and Sobcszyk, 1984). The fiducial derivative $d_a\omega(b)$ is given by

$$d_a\omega(b) = \tfrac{1}{2} d_a\omega_{ij}(b)\gamma^i \wedge \gamma^j \tag{36}$$

with

$$d_a\omega_{ij}(b) = a \cdot \mathbf{D}\omega_{ij}(b) - \omega_{ij}(b \cdot \mathbf{D}a) \tag{37}$$

These two relations enable us to compute the two first terms of (35), i.e.,

$$d_a\omega(b)-d_b\omega(a)=\tfrac{1}{2}(a\cdot\mathbf{D}\omega_{ij}(b)-b\cdot\mathbf{D}\omega_{ij}(a))\gamma^i\wedge\gamma^j-\omega([a,b]) \quad (38)$$

with $[a,b]=a\cdot\mathbf{D}b-b\cdot\mathbf{D}a$ the ordinary commutator of a and b.

If we take $a=\gamma_k$ and $b=\gamma_l$; then (37) becomes

$$\begin{aligned} d\gamma_k\,\omega(\gamma_l)&-d\gamma_l\,\omega(\gamma_k)\\ &=\tfrac{1}{2}(\gamma_k\cdot\mathbf{D}\omega_{ij}(\gamma_l)-\gamma_l\cdot\mathbf{D}\omega_{ij}(\gamma_k))\gamma^i\wedge\gamma^j-\omega([\gamma_k,\gamma_l]) \end{aligned} \quad (39)$$

The operator $\gamma_k\cdot\mathbf{D}$ can be written

$$\gamma_k\cdot\mathbf{D}=\gamma_k\cdot h_j^{\ i}\gamma^j\partial_i=h_j^{\ i}\delta_k^{\ j}\partial_i=h_k^{\ i}\partial_i=X_k \quad (40)$$

and the commutator of γ_k and γ_l is

$$[\gamma_k,\gamma_l]=C^m_{\ kl}\gamma_m$$

The linearity of ω enables us to compute explicitly the quantity $\omega([\gamma^k,\gamma^l])$:

$$\begin{aligned}\omega([\gamma_k,\gamma_l])&=\omega(C^m_{\ kl}\gamma_m)=C^m_{\ kl}\omega(\gamma_m)\\ &=C^m_{\ kl}\tfrac{1}{2}\omega_{ij}(\gamma_m)\gamma^i\wedge\gamma^j\end{aligned} \quad (41)$$

Putting (40) and (41) into (39) gives

$$d\gamma_k\,\omega(\gamma_l)-d\gamma_l\,\omega(\gamma_k)=\tfrac{1}{2}H_{ij}(\gamma_k,\gamma_l)\gamma^i\wedge\gamma^j \quad (42)$$

with

$$H_{ij}(\gamma_k,\gamma_l)=X_k\omega_{ij}(\gamma_l)-X_l\omega_{ij}(\gamma_k)-C^m_{\ kl}\omega_{ij}(\gamma_m) \quad (43)$$

We easily see that $H_{ij}(\gamma_k,\gamma_l)$ is antisymmetric in the indices i, j and k, l, so that the quantity (42) can be written

$$d\gamma_k\,\omega(\gamma_l)-d\gamma_l\,\omega(\gamma_k)=\sum_{i<j}H_{ij}(\gamma_k,\gamma_l)\gamma^i\wedge\gamma^j \quad (44)$$

To compute explicitly the last term of (34), it is interesting to use the general formula giving the cross product of two 2-vectors:

$$\begin{aligned}(a\wedge b)\times(c\wedge d)&=a\wedge(b\cdot(c\wedge d))-b\wedge(a\cdot(c\wedge d))\\ &=(b\cdot c)a\wedge d-(b\cdot d)a\wedge c\\ &\quad+(a\cdot d)b\wedge c-(a\cdot c)b\wedge d\end{aligned} \quad (45)$$

Therefore

$$\omega(a)\times\omega(b)=\tfrac{1}{2}\omega_{ij}(a)\omega_{kl}(b)(\gamma^i\wedge\gamma^j)\times(\gamma^k\wedge\gamma^l) \quad (46)$$

Using (45), we find

$$\begin{aligned}(\gamma^i\wedge\gamma^j)\times(\gamma^k\wedge\gamma^l)&=\eta^{jk}(\gamma^i\wedge\gamma^l)-\eta^{jl}(\gamma^i\wedge\gamma^k)\\ &\quad-\eta^{ik}(\gamma^j\wedge\gamma^l)+\eta^{il}(\gamma^j\wedge\gamma^k)\end{aligned} \quad (47)$$

Putting (47) into (46), we can then write the product $\omega(a)\times\omega(b)$ as the 2-vector

$$\omega(a)\times\omega(b)=L_{il}\gamma^i\wedge\gamma^l \tag{48}$$

with

$$L_{il}=\eta^{jk}\omega_{ij}(a)\omega_{kl}(b)$$

Since the product $\gamma^i\wedge\gamma^l$ is antisymmetric, it is only the antisymmetric part of L_{il} that contributes to (48). Therefore we can write

$$\omega(a)\times\omega(b)=\tfrac{1}{2}K_{il}\gamma^i\wedge\gamma^l \tag{49}$$

with

$$K_{il}=L_{il}-L_{li}=\eta^{jk}\omega_{ij}(a)\omega_{kl}(b)-\eta^{jk}\omega_{lj}(a)\omega_{ki}(b) \tag{50}$$

and (49) becomes

$$\omega(a)\times\omega(b)=\sum_{i<j}K_{ij}\gamma^i\wedge\gamma^j \tag{51}$$

Putting the results obtained in (44) and (51) into (34), we are now able to write the curvature 2-vector:

$$R(\gamma_k\wedge\gamma_l)=\sum_{i<j}[H_{ij}(\gamma_k,\gamma_l)+K_{ij}(\gamma_k,\gamma_l)]\gamma^i\wedge\gamma^j \tag{52}$$

This quantity corresponds exactly to the curvature 2-form given by the second Cartan structural equation (33′). The components R_{ijkl} of the Riemann tensor are the coefficients of $\gamma^i\wedge\gamma^j$ in $R(\gamma_k\wedge\gamma_l)$, which can therefore be written

$$R(\gamma_k\wedge\gamma_l)=\sum_{i<j}R_{ijkl}\gamma^i\wedge\gamma^j \tag{53}$$

The symmetries of R_{ijkl} come of course from the symmetries of the quantities H_{ij} and K_{ij}.

5. CONCLUSIONS

There are essentially two common methods to compute the fundamental quantities in differential geometry. The first is the tensorial method, which starts with the components of the metric tensor g_{ij} and/or the components of a tetrad and consists in the direct computation of the components of the Christoffel symbols, Riemann tensor, Ricci, scalar curvature, and so on.

This kind of computation has been widely explored in various algebraic programming languages, such as REDUCE, MACSYMA (with the package CTENSR), SHEEP, and STENSOR (which is specialized in indices manipulation).[2]

The second method is based on exterior calculus or Cartan calculus. This method is easier than the tensorial one when we want to make calculations by hand, but this characteristic does not seem to have been exploited in algebraic programs (at present we only know of the program EXCALC written by E. Schruefer that tries to achieve this goal). One of the fundamental operations of exterior calculus is the exterior derivative. It is a well-defined, base-independent operation, which is translated into any programming language as a procedure. The approach in geometric calculus is quite different, in the sense that we can define as many differential operators we want because they are introduced as vectors with possibly complicated components (see, for example **D** in this paper) and the operation of differentiation performed depends on the product chosen between the vector-operator and the quantity we act upon ($\mathbf{d} \wedge v$ is the curl or exterior derivative and $\mathbf{d} \cdot v$ is the divergence, for instance). As there are many different products in geometric calculus (Hestenes and Sobcszy, 1984), there are many different derivatives. For example, in this paper the components of the operator **D** given by (4) have been chosen to deal more easily with the problem of moving frames.

It seems to us that geometric calculus based on Clifford calculus provides us with more possibilities than Cartan's calculus because it manipulates multivectors, which are more general quantities than differential forms. Furthermore, it is easy to do exterior calculus with geometric calculus, but the converse is not generally true.

All the quantities introduced in this paper have been computed in the particular case of the Kerr metric (and some others) with a package written in MACSYMA doing general manipulations on multivectors [a simpler application on the Schwarzschild metric can be found in (Moussiaux and Tombal, 1987)]. It is difficult to make a valid comparison of execution times between Cartan and geometric calculus for particular quantities because the methods are quite different, but globally the geometric method is faster by a factor of two. We could say that the advantages largely depend on the choice of the quantities we want: if we need, for example, only the component Γ_{103} of the connection, then it is more effective to use the relation (24). Conversely, if we need the Γ_{ij3} for all i and j, then it is better to

[2]REDUCE: *User's Manual,* Antony C. Hearn; SHEEP: I. Frick, Institute of Theoretical Physics, University of Stockholm; STENSOR: L. Hornfeld, University of Stockholm; MACSYMA: *Reference Manual,* Symbolics, Inc., MIT Cambridge, Massachusetts; EXCALC: *User's Manual,* Eberhard Schruefer, 1986.

compute the 2-vector $\omega(\gamma_3)$ than (24) for all i and j. The same remark applies for other quantities, such as the Riemann tensor.

Another aspect of geometric calculus that seems promising is the possibility to compute quantities that do not exist or are less easily computed with another method. The concept of "coderivative" of a vector, for example, when generalized to any multivector as shown by the formula $\delta_a A = d_a A + \omega(a) \times A$ will certainly provide us with a powerful technique for computing covariant derivatives.

In a forthcoming paper we will extend the theory given here to the more general case where the metric η_{ij} is no longer constant, as in the case, for example, of homogeneous cosmological models.

ACKNOWLEDGMENTS

We would like to thank Prof. D. Hestenes for his interest and for having answered so generously all our questions.

REFERENCES

Hestenes, D., and Sobcszyk, G. (1984). *Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht.

Hestenes, D. (1986). *Space-Time Algebra*, Gordon and Breach, New York.

Hestenes, D. (1986). Curvature calculations with spacetime algebra, *International Journal of Theoretical Physics*, **25**, 581–588.

Moussiaux, A., and Tombal, Ph. (1987). Curvature calculations with GEOCAL, *International Journal of Theoretical Physics* (in press).

Tombal, Ph., and Moussiaux, A. (1986). Algebraic programming of geometric calculus and Clifford algebra, *International Journal of Symbolic Computation.*

Von Westenholz, C. (1978). *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam.